An Analysis of the Anomalies in Traditional Discounting Models

Salvador Cruz Rambaud* **y María José Muñoz Torrecillas**

Universidad de Almería. España

ABSTRACT

The discounting utility model (DU model), introduced by Samuelson in 1937, has dominated the economic analysis of intertemporal choice being, along with the expected utility model (EU model), one of the widely used traditional discounting models. Nevertheless, several empirical studies, mainly arisen from the field of psychology, have described the individual behavior when discounting real or hypothetical rewards, showing the existence of "anomalies" or violations of the traditional discounting models (DU and EU) axioms. These "anomalies in intertemporal choice" have been labelled as: delay effect, magnitude effect, sign effect, sequence effect, delay-speedup asymmetry and spreading effect. Even hyperbolic discounting has been considered, by some authors, as an anomaly of intertemporal choice. In this paper, first we will describe these anomalies and how the experiments in the field of psychology have detected them. Starting from this empirical evidence, our aim is to include the effect of these anomalies in the mathematical expression of the discount functions describing the intertemporal choice of individuals. The empirical application of the traditional discounting models has shown the aforementioned anomalies which set the stage for a change in normative theory and for the searching of new discounting models.
Key words: Intertemporal choice, anomalies, discount function, discount rate.

RESUMEN

Análisis de las anomalías en los modelos tradicionales de descuento. El modelo de utilidad descontada (modelo DU), introducido por Samuelson en 1937, ha dominado los análisis económicos sobre elección intertemporal, siendo, junto con el modelo de utilidad esperada (modelo EU), uno de los modelos tradicionales de descuento cuyo uso ha sido generalizado. No obstante, diversos estudios empíricos, en su mayoría realizados en el ámbito de la psicología, han descrito la conducta de los individuos al descontar recompensas reales o hipotéticas, poniendo de manifiesto la existencia de "anomalías" o violaciones de los axiomas de esos modelos tradicionales de descuento (DU y EU). Estas "anomalías en la elección intertemporal" han sido denominadas: efecto plazo, efecto magnitud, efecto signo, efecto secuencia, efecto asimetría respecto al aplazamiento-anticipación y efecto diseminación. Incluso el descuento hiperbólico ha sido clasificado por algunos autores como una anomalía de la elección intertemporal. En este artículo describiremos, en primer lugar, esas anomalías y cómo se han detectado en los experimentos

* Reprints may be obtained from the first author: Facultad de Universidad de Almería (España). 04071, Almería. E-mail: scruz@ual.es

realizados dentro del ámbito de la psicología. A partir de esta evidencia empírica, nuestro objetivo es incluir el efecto de dichas anomalías en la expresión matemática de las funciones de descuento que describen la elección intertemporal de los individuos. El descubrimiento de la evidencia empírica de anomalías en los modelos tradicionales de descuento abre un camino hacia el cambio en la teoría normativa y hacia la búsqueda de nuevos modelos de descuento.
Palabras clave: Elección intertemporal, anomalías, ley financiera de descuento, factor de descuento.

Traditionally, psychology and economy have used the scientific method to explain human behavior, differing, however, in their respective approaches. Psychology starts from the empirical analysis of particular cases and then develops a theory whose validity is tested with other observations. Economy, however, is more theory-based, starting from a theoretical approach that is then used for a wide range of applications.

These methodological differences also arise when revising the economic and psychological literature about intertemporal choice, which is the subject we are interested in. Basic research on intertemporal choice made by economists has been focused on the discounted utility model, testing the validity of this model and its implications. Nevertheless, basic research on intertemporal choice by psychologists has been focused on different questions. “Some have been concerned with measuring individual differences in the propensity to delay gratification, others with situational determinants of impulsivity, and still others with cognitive and emotional mechanisms underlying intertemporal choice” (Loewenstein *et al.*, 2003).

In this paper we will focus on a series of anomalies of the discounted utility model –traditionally applied by economists to intertemporal choice– arising from the empirical observation of several intertemporal choice behavior, mainly in the psychological field. First we will describe these anomalies and how the experiments in the field of psychology have led to their finding. Starting from this empirical evidence, we will establish the effect of these anomalies in the expression of the discount functions describing the intertemporal choice of individuals. Our aim is to achieve one or several discounting models depending on the particular cases and including the anomalies observed in the intertemporal choice behavior.

Many empirical studies try to find the mathematical function that best fits the individuals’ behavior when discounting future rewards. Discounting the value of future rewards may well be an adaptive response to the risks associated with waiting for delayed rewards (Kagel *et al.*, 1986). After all, as the delay to an outcome increases, the probability of receiving that outcome usually decreases (Green & Myerson, 1996).

From a temporal discounting perspective, the interesting issue is the nature of the mathematical relation between delay and value. From the economic and psychological fields, two different approaches have usually been employed to determine this function. Economists have taken a “rational” approach to the problem and have attempted to derive a formula from theoretical assumptions, often based on normative models of what organisms ought to do. In contrast, psychologists have taken an “empirical” approach and have attempted to find the formula that best describes what organisms are observed to do (i.e., Myerson & Green, 1995).

The standard economic model of discounted utility (Samuelson, 1937) is one of the proposed formulae to describe temporal discounting. This model supposes that the value of a future reward is discounted because of the risk that waiting for its reception implies. Given a contingent relationship between the choice of a reward and its eventual reception, it is supposed that a constant hazard rate exists in this relation. If there is a constant hazard rate associated with waiting, then the temporal discounting function will be exponential (Myerson & Green, 1995).

Recently, an increasing number of studies have replaced the constant discount (standard) and the exponential discounting function with the hyperbolic discounting function. Empirical studies of discounting suggest that people discount the future at hyperbolic rather than exponential rates, or more generally that they discount the distant future at lower rates than they discount the near future (Azfar, 1999). This behavior has sometimes been termed as an anomaly or paradox for rational choice theory (Loewenstein & Thaler, 1989).

Adjusted to the same data, the hyperbola will initially (at short delays) decrease faster than the exponential function, but will (at long delays) decrease more slowly than the exponential (Myerson & Green, 1995). The hyperbola has been initially justified on empirical grounds (Ainslie, 1992; Mazur, 1987; Rachlin, 1989; Rachlin, Raineri & Cross, 1991; Rodríguez & Logue, 1988) as a variation on the formula that implies raising the denominator to a power (Green, Fry & Myerson, 1994; Loewenstein & Prelec, 1992).

Most of these contributions have appeared in the behavioral psychology field and, although the obtained results involve decisional processes that imply the discount of delayed rewards, they can also be applied to the economic field to discount the flows from investment projects. Laibson *et al.* (1998) and Angeletos *et al.* (2001) study decisional models of consumption-saving using exponential and hyperbolic discounting and showing how the last can better explain several empirical observations, including some anomalies from traditional discounting models (discounting utility and expected utility models).

The result of the empirical work from Green & Myerson (1996) points out the hyperbola-like discounting as the model that best explains temporal discounting both at individual and at group level. The same conclusion has been obtained from their previous work (Myerson & Green, 1995). Apart from the above mentioned works, there are many studies that propose hyperbolic or hyperbola-like discounting functions: Azfar (1999), Ainslie (1975), Green, Myerson & Ostazewski (1999), Harvey (1986), Henderson & Bateman (1995), Herrnstein (1981), Kirby (1997), Kirby & Marakovic (1995), Laibson (1997), Loewenstein & Elster (1992); Mazur (1987), Myerson, Green & Warusawitharana (2001), Prelec (1989), and Richards *et al.* (1997).

Discounted utility models (DU models) and expected utility models (EU models), parallel in their structure, consider decision-makers facing the selection of alternatives based on the weighted addition of utilities, being these weights either probabilities (for the EU model) or discount factors based on temporal delays (for the DU model). More precisely, the discounted utility theory states that individuals discount future events at a constant rate, so the value of an experience during a period of time [0,*T*] is given by:

$$U_0 = \sum_{t=0}^{T} \delta^t u_t$$

where U_0 is the present value of the experience, u_t is the utility obtained from the experience at moment *t*, and δ is the discount factor, whose value is supposed to be less than one (corresponding with a positive time preference).

This model of discounted utility, first proposed by Samuelson in 1937, has been adopted by economists and theorists of decision, without discussion. This quick acceptance, as well as the received one for the expected utility model, is due to its formal simplicity and its similarity to the already well-known financial calculus systems of actuarial and present values. These models were completed, clarifying its logic and its fundamental assumptions, by von Neumann & Morgenstern (1953) and Koopmans (1960) who worked on EU and DU models, respectively. It was already in the nineteen-eighties with the publication of Thaler's work "Some empirical evidence on dynamic inconsistency" (1981), when the critiques to this model started. Even Samuelson said that it was not a particularly realistic model of how people make intertemporal choices. Thus, empirical studies on intertemporal choice arise, illustrating the so-called "intertemporal choice anomalies or inconsistencies": Thaler (1981), Christensen-Szalanski (1984), Benzion, Rapaport & Yagil (1989), Loewenstein & Prelec (1991, 1992), Kirby & Marakovic (1995), Prelec & Loewenstein (1991), Green, Fristoe & Myerson (1994), and Kirby & Herrnstein (1995), Chapman (1996, 2000, 2001). All of them reach similar conclusions about some guidelines (anomalies) in the discount rates that appear in their experiments on intertemporal choice and do not fit the predictions of the traditional discounting models, particularly of the DU model. These experiments have employed hypothetical and real monetary rewards. And, in recent years, investigations have used not only monetary choices, but also choices in other domains, most notably health (Chapman, 2001).

The anomalies have been labelled as delay effect, magnitude effect, sign effect, sequence effect, spreading effect and delay-speedup asymmetry. Even hyperbolic discounting is included in this range of intertemporal choice anomalies (Loewenstein & Thaler, 1989; Read & Loewenstein, 2000). In the following sections of this paper we explain these anomalies or violations of DU and EU axioms, revising the empirical studies leading to each one of these anomalies, their conclusions and their implications in the expression of the discount functions, as well as in the discount rate estimates. Finally, the last section summarizes and concludes.

The delay effect

Several authors have included, in their studies of the discounted utility model, some failures when fitting the empirical data to this model, that they have labelled as "anomalies". Among them we can find the delay effect consisting of the decrease of the discount rate as waiting time increases, that is, the discount rates tend to be higher in short intervals than in longer ones. This effect has been shown for both monetary decisions (Benzion *et al.*, 1989; Thaler, 1981) and non-monetary decisions (Christensen-Szalanski, 1984; Chapman, 2001; Thaler, 1981).

Thus, for example, in the empirical study of Benzion *et al.* (1989), the discount rates for deferring a 200-dollar amount are deduced, obtaining higher average discount rates for shorter time intervals: 0.428; 0.255; 0.230 and 0.195 for delays of 6 months, 1, 2, and 4 years, respectively.

Delay effect can derive in preference reversals (Green, Fristoe & Myerson, 1994; Kirby & Herrnstein, 1995), whose modelling can be obtained with a hyperbolic discount function better than the exponential one specified by the normative theory (Kirby & Marakovic, 1995). In effect, unlike exponential discount functions, hyperbolic functions can intersect for higher and lower rewards, indicating preference reversals. Christensen-Szalanski (1984) showed such a preference reversal in women deciding whether to have anaesthesia for childbirth. As the moment of childbirth approached, the women showed a stronger preference for the immediate relief, even they shift their preferences from avoiding using anaesthesia toward using it to avoid pain.

An alternative consideration to delay effect is subadditive discounting whereby the discount in a long time interval is bigger when the delay is subdivided. This can explain the delay effect (higher discount rates for shorter intervals) but not preference reversals (Christensen-Szalanski, 1984; Green *et al.*, 1994; Kirby & Herrnstein, 1995).

Prelec & Loewenstein (1991) set out delay effect in a similar way of hyperbolic discounting and propose the property of *decreasing absolute sensibility*. It means that, for example, the difference between years 0 and 2 seems greater than the difference between years 6 and 8. They call this anomaly *common difference effect* and *immediacy effect*.

The common difference effect implies that the impact of a constant difference of time between two results becomes less significant as the two results are more distant in time. Thus, for example, a person will be indifferent between having 20 euros today and having 25 euros in one month, but will prefer 25 euros in 11 months to 20 euros in 10 months. Or, if we explain the example with non-monetary results, as Thaler (1981) presents in his empirical study, a person can prefer an apple today to two apples tomorrow, but at the same time he will prefer two apples in 51 days to one apple in 50 days.

We can set out this effect as follows:

$$(x,s) \sim (y,t)\text{, but } (x,s+h) \prec (y,t+h)\text{, for } y > x,\ s < t \text{ and } h > 0\text{.}$$

That is to say, if two capitals (x,s) and (y,t) are indifferent, $(x,s) \sim (y,t)$, their projections onto a common instant *p* (usually, *p* is taken as 0) have to coincide using the mathematical criterion represented by a discount function:

$$xA(s,p) = yA(t,p) \text{ if and only if } \frac{x}{y} = \frac{A(t,p)}{A(s,p)} = v(s,t,p)\text{,}$$

being $A(t,p)$ the discount function which represents the amount available at *p* instead of one euro available at *t*, and $v(s,t,p)$ the corresponding financial factor.

In the same way, if $(x,s+h) \prec (y,t+h)$, the relationship between the amounts with the discount function will be:

$$xA(s+h,p) < yA(t+h,p) \text{ if and only if } \frac{x}{y} < \frac{A(t+h,p)}{A(s+h,p)} = v(s+h,t+h,p).$$

Then:

$$v(s,t,p) < v(s+h,t+h,p),$$

which is consistent with the following definition that includes the concept of diminishing (increasing) impatience.

Definition 1. The financial factor associated to a discount function $A(t,p)$ is *increasing* (resp. *decreasing*) if

$$v(s,t,p) \leq v(s+h,t+h,p), \quad h > 0$$

$$(\text{resp. } v(s,t,p) \geq v(s+h,t+h,p), \quad h > 0).$$

Theorem 1. A necessary and sufficient condition for a financial factor being increasing is that the discount rate will be decreasing.

Proof. First, we will show that the condition is necessary. As the financial factor is increasing, for all $p \leq s \leq t$ it is verified that:

$$v(s,s+h,p) \leq v(t,t+h,p), \quad h > 0,$$

$$\frac{A(s+h,p)}{A(s,p)} \leq \frac{A(t+h,p)}{A(t,p)},$$

$$\frac{1-\frac{A(s+h,p)}{A(s,p)}}{h} \geq \frac{1-\frac{A(t+h,p)}{A(t,p)}}{h}.$$

Taking limits when h tends to 0:

$$-\left.\frac{\partial \ln A(x,p)}{\partial x}\right|_{x=s} \geq -\left.\frac{\partial \ln A(x,p)}{\partial x}\right|_{x=t},$$

so, the discount rate is decreasing.

Let us see now that the condition is sufficient. We will start from $v(s,t,p)$, with $p < s < t$ and $h > 0$.

$$v(s,t,p) = e^{-\int_s^t \delta(x,p)dx} \leq e^{-\int_s^t \delta(x+h,p)dx} = e^{-\int_{s+h}^{t+h} \delta(x,p)dx} = v(s+h,t+h,p),$$

so, the financial factor is increasing.

Some researchers have argued that, besides common difference effect, a discontinuity of preference appears, in fact, when time dimension approaches to its maximal importance point (that is, $t = 0$). The *immediacy effect* means that decision-makers give special importance to the immediate results, that is to say:

$(x,s) \sim (y,t)$ implies $(x,s+h) \prec (y,t+h)$, for $t=0$ and $y>x$, $h>0$.

This can be seen in the extremely high discount rates estimated for short delays in several empirical studies about discounting (Thaler, 1981; Benzion, Rapaport & Yagil, 1989) and it can also be observed in intertemporal decisions not implying monetary payments (Christensen-Szalanski, 1984).

The immediacy effect is a special case of stationarity violation and, therefore, it can formally be included inside the common difference effect. Several researchers, however, think that these phenomena are qualitatively different and justify a separate treatment (Prelec & Loewenstein, 1991). It could be formally explained as follows:

If we consider the equivalent capitals (x,p) and (y,t), being $y>x$ and $p<t$, we can state the following relationship between the amounts with the discount function:

$$xA(p,p) = yA(t,p) \text{ if and only if } \frac{x}{y} = A(t,p), \text{ since } A(p,p)=1.$$

Increasing the appraisal instant in a constant $h>0$, there will be a more preferred capital, $(x,p+h) \prec (y,t+h)$, and the relationship between the amounts with the discount functions will be:

$$xA(p+h,p) < yA(t+h,p) \text{ if and only if } \frac{x}{y} < \frac{A(t+h,p)}{A(p+h,p)} = v(p+h,t+h,p),$$

and so, we can conclude that:

$$A(t,p) < v(p+h,t+h,p).$$

This means that the average discount rate *spot* in the interval $\lfloor p,t \rfloor$ is lower than the average discount rate *forward* at p for the interval $\lfloor p+h,t+h \rfloor$.

We can also consider that the appraisal instant, *p*, is variable; then the discounting function will be contractive, implying a decreasing instantaneous discount rate in the direction of the vector (1,1).

Definition 2. A discounting function $A(t,p)$ is *contractive* (resp. *expansive*) if

$$A(t,p) \leq A(t+h,p+h), \quad h>0$$

$$\text{(resp. } A(t,p) \geq A(t+h,p+h), \quad h>0 \text{).}$$

<u>*Theorem 2*</u>. A sufficient condition for an expansive (resp. contractive) discounting function is that the instantaneous discount rate is increasing (resp. decreasing) in the direction of the vector (1,1).

Proof. In effect, for all $h>0$,

$$A(t+h,p+h) = e^{-\int_{p+h}^{t+h} \delta(x,p+h)dx} = e^{-\int_{p}^{t} \delta(x+h,p+h)dx} \leq e^{-\int_{p}^{t} \delta(x,p)dx} = A(t,p).$$

It can be shown, however, that this condition is not necessary.

<u>*Theorem 3*</u>. A sufficient condition for a subadditive discount function[1] is that the corresponding financial factor is increasing and the discounting function expansive.

Proof. Starting from the financial factor associated to the discounting function at p for the interval $[s, t]$ (then, $p \leq s \leq t$):

$$v(s,t,p) = \frac{A(t,p)}{A(s,p)} \geq$$

(as $p - s \leq 0$ and the financial factor is increasing)

$$\geq \frac{A(t+p-s,p)}{A(s+p-s,p)} = \frac{A(t+p-s,p)}{A(p,p)} = \frac{A(t+p-s,p)}{1} = A(t+p-s,p) \geq$$

(as $-p + s \geq 0$ and A is expansive)

$$\geq A(t+p-s-p+s, p-p+s) = A(t,s),$$

that is,

$$\frac{A(t,p)}{A(s,p)} \geq A(t,s),$$

from where we can deduce that

$$A(t,p) \geq A(t,s) \cdot A(s,p)$$

and, so the discounting function is subadditive.

Summarizing, the previously revised researches on intertemporal choice, have shown that the discount rate implicit in choices will vary inversely with the length of time to be waited for. The experiments have been conducted mainly in the psycological field, making people to decide between real or hypothetical options of delayed monetary or even non-monetary rewards.

Other related but different phenomenon is subadditive discounting that implies higher discount when the delay is subdivided. In theorems 1 to 3 we have set out the conditions for a discounting function including both delay effect and subadditive discounting.

The magnitude effect

Another anomaly of the discounted utility model is the so-called magnitude effect that implies a higher discount for smaller amounts than for bigger ones and that it is performed when the discount function is non-homogeneous, that is to say, the discount function depends on the amount c. So the subjective discount rates vary not only with the period until obtaining the reward, but also with the magnitude of the result or reward. Smaller rewards tend to result in higher discount rates. That way, a subject will prefer 100 euros now to 150 in one year, but will also prefer 15,000 euros in one year to 10,000 euros now; however both choices give a 50% profit for a one year wait.

We can observe the magnitude effect in the empirical studies of Thaler (1981), Benzion *et al.* (1989) and Holcomb & Nelson (1989) who worked with real monetary results. More precisely, in the first one, the subjects were indifferent, on average, between 15 dollars immediately and 60 dollars in one year, 250 dollars immediately and 350 in one year, and 3,000 dollars immediately and 4,000 in one year, implying discount rates of 75%, 29% and 25%, respectively. So we can observe how larger amounts are discounted at a lower rate than smaller amounts.

Let us suppose that the instantaneous discount rate is inversely proportional to the discounted amount:

$$\delta(z) = \frac{k}{c}, \text{ with } k = 100.$$

In this case, the respective discount function is:

$$A(c,z) = c \cdot e^{-\int_0^z \frac{k}{c}dx} = c \cdot e^{-\frac{k}{c}z}.$$

Considering, for example, the capitals (100;0); (150;1); (10,000;0) and (15,000;1), the magnitude effect is verified, since:

$$A(100;0) = 100 \text{ ¤}$$
$$A(150;1) = 77.01 \text{ ¤}$$
$$A(10{,}000;0) = 10{,}000 \text{ ¤}$$
$$A(15{,}000;1) = 14{,}900.33 \text{ ¤}$$

from which:

$$(100,0) \succ (150,1)$$

and

$$(10{,}000;0) \prec (15{,}000;1).$$

Prelec & Loewenstein (1991) formulate the magnitude effect as follows:

$$(x,s) \sim (y,t) \text{ implies } (\alpha x,s) \prec (\alpha y,t), \text{ for } y > x > 0 \text{ and } s < t$$

and

$$(-x,s) \sim (-y,t) \text{ implies } (-\alpha x,s) \succ (-\alpha y,t)$$

and they propose, in order to explain it, the property of *proportional increasing sensibility*: if we increase the absolute magnitude of all attribute values by a common multiplicative constant, the attribute weight will increase. More precisely, if:

$$(a_1,b_1) \sim (a_2,b_2) \text{ and } \alpha a_1 > 0, |\alpha| > 1, \text{ then}$$

$$(\alpha a_1,b_1) \prec (\alpha a_2,b_2) \text{ if and only if } (\alpha a_1,b_2) \prec (\alpha a_2,b_1).$$

We can enunciate the following theorems about the magnitude effect that state some consequences on the discounting function.

<u>*Theorem 4*</u>. If the magnitude effect is verified, for all *x, y, s* and *t*, such that $x < y$ and $s < t$, if $(x,s) \sim_p (y,t)$ then $(\alpha x,s) \succ_p (\alpha y,t)$, for all α between 0 and 1.

Proof. Let us suppose that $(\alpha x,s) \preceq_p (\alpha y,t)$, being $0 < \alpha < 1$. Then we can find

$y' \leq y$ such that:

$$(\alpha x, s) \sim_p (\alpha y', t) \ .$$

As the magnitude effect is verified and $\frac{1}{\alpha} > 1$,

$$\left(\frac{1}{\alpha}\alpha x, s\right) \prec_p \left(\frac{1}{\alpha}\alpha y', t\right),$$

so:

$$\left(\frac{1}{\alpha}\alpha x, s\right) \prec_p \left(\frac{1}{\alpha}\alpha y', t\right) \preceq (y, t) \ ,$$

from which:

$$(x, s) \prec_p (y, t) \ ,$$

in contradiction with the fact of being $(x, s) \sim_p (y, t)$.

<u>Theorem 5</u>. A necessary condition for the magnitude effect is that the underlying discount function is subadditive with respect to the amount.

Proof. Indeed, for all *x, y, s* and *t*, such that $x < y$ and $s < t$, if $(x, s) \sim_p (y, t)$ then $(\alpha x, s) \prec_p (\alpha y, t)$, for all $\alpha > 1$. In particular, if $s = p$ and $(x, p) \sim_p (y, t)$, then:

$$A(x, p, p) = A(y, t, p) \ .$$

Taking into account that $A(x, p, p) = x$, it would remain:

$$x = A(y, t, p) \ .$$

On the other hand, because of the magnitude effect,

$$(\alpha x, p) \prec_p (\alpha y, t) \ ,$$

which implies that:

$$A(\alpha x, p, p) < A(\alpha y, t, p) \ .$$

Since $A(\alpha x, p, p) = \alpha x$,

$$\alpha x < A(\alpha y, t, p)$$

and, substituting *x* by its value,

$$\alpha A(y, t, p) < A(\alpha y, t, p),$$

from which it is deduced that *A* is subadditive with respect to the amount.

The inverse implication is not true, since, for example, any discount function of the form:

$$A(c, t, p) = c^2 f(t, p), \ \ c > 0$$

is subadditive, but does not verify the magnitude effect.

In effect,

$$A(x + y, t, p) = (x + y)^2 f(t, p) > x^2 f(t, p) + y^2 f(t, p) = A(x, t, p) + A(y, t, p),$$

so, *A* is subadditive.

However, if $(x,s) \sim_p (y,t)$, then:

$$A(x,s,p) = A(y,t,p) \ ,$$

from which:

$$x^2 f(s,p) = y^2 f(t,p) \ .$$

Obviously, for all α,

$$\alpha^2 x^2 f(s,p) = \alpha^2 y^2 f(t,p) \ ,$$

from where:

$$(\alpha x, s, p) \sim_p (\alpha y, t, p) \ ,$$

and so the magnitude effect is not verified.

Theorem 6. A necessary and sufficient condition for the magnitude effect is that for all *x*, *y*, *s* and *t*, such that $x < y$ and $s < t$, if $(x,s) \sim_p (y,t)$ then the following relationship between the directional derivatives is verified:

$$D_{(x,0,0)} A(c,s,p) < D_{(y,0,0)} A(c,t,p) \ .$$

Proof. First, let us see that the condition is necessary. In effect, for all *x*, *y*, *s* and *t*, such that $x < y$ and $s < t$, if $(x,s) \sim_p (y,t)$ then $(\alpha x, s) \prec_p (\alpha y, t)$, for all $\alpha > 1$. This way, it is verified that:

$$A(x,s,p) = A(y,t,p)$$

and:

$$A(\alpha x, s, p) < A(\alpha y, t, p) \ ,$$

from which:

$$\frac{A(\alpha x,s,p)}{A(\alpha y,t,p)} < \frac{A(x,s,p)}{A(y,t,p)} \ ,$$

and so:

$$\frac{A(\alpha x,s,p) - A(x,s,p)}{A(\alpha y,t,p) - A(y,t,p)} < \frac{A(x,s,p)}{A(y,t,p)} = 1 \ .$$

As $\alpha > 1$, $\alpha = 1 + h$, with $h > 0$,

$$\frac{A(x+hx,s,p) - A(x,s,p)}{A(y+hy,t,p) - A(y,t,p)} < 1 \ ,$$

from which:

$$\frac{\dfrac{A(x+hx,s,p) - A(x,s,p)}{hx}}{\dfrac{A(y+hy,t,p) - A(y,t,p)}{hy}} < \frac{y}{x} \ .$$

Taking limits when $h \to 0$, it is verified that:

$$\frac{\left.\frac{\partial A(c,s,p)}{\partial c}\right|_{c=x}}{\left.\frac{\partial A(c,t,p)}{\partial c}\right|_{c=y}} < \frac{y}{x},$$

or, that is the same,

$$x\left.\frac{\partial A(c,s,p)}{\partial c}\right|_{c=x} < y\left.\frac{\partial A(c,t,p)}{\partial c}\right|_{c=y},$$

from where it is deduced that:

$$D_{(x,0,0)}A(c,s,p) < D_{(y,0,0)}A(c,t,p).$$

Next, let us see that the condition is sufficient. In effect, suppose that

$$D_{(x,0,0)}A(c,s,p) < D_{(y,0,0)}A(c,t,p)$$

and that for all *x, y, s* and *t*, such that $x < y$ and $s < t$, $(x,s) \sim_p (y,t)$. We want to show that $(\alpha x, s) \prec_p (\alpha y, t)$, for all $\alpha > 1$, that is to say:

$$A(\alpha x, s, p) < A(\alpha y, t, p).$$

Taking into account the inequality between the directional derivatives, there is a $\alpha_1 > 1$ such that

$$A(\alpha_1 x, s, p) < A(\alpha_1 y, t, p).$$

If $\alpha_1 \geq \alpha$, the theorem is already shown. If $\alpha_1 < \alpha$, we can find a $y_1 < y$ such that:

$$A(\alpha_1 x, s, p) < A(\alpha_1 y_1, t, p),$$

which implies that:

$$D_{(\alpha_1 x,0,0)}A(\alpha_1 c, s, p) < D_{(\alpha_1 y,0,0)}A(\alpha_1 c, t, p).$$

Repeating again the previous reasoning, there is a $\alpha_2 > 1$ such that

$$A(\alpha_2 \alpha_1 x, s, p) < A(\alpha_2 \alpha_1 y_1, t, p).$$

If $\alpha_2 \alpha_1 \geq \alpha$, the theorem is already shown. If $\alpha_2 \alpha_1 < \alpha$, we can find a $y_2 < y_1$ such that:

$$A(\alpha_2 \alpha_1 x, s, p) = A(\alpha_2 \alpha_1 y_2, t, p).$$

The reasoning can be repeated so many times as we want, up to finding a $\alpha_n > 1$ such that $\alpha_n \ldots \alpha_2 \alpha_1 > \alpha$ and

$$A(\alpha_n \ldots \alpha_2 \alpha_1 x, s, p) < A(\alpha_n \ldots \alpha_2 \alpha_1 y_{n-1}, t, p),$$

which also implies the inequality with α:

$$A(\alpha x, s, p) < A(\alpha y_{n-1}, t, p).$$

As $y_{n-1} < \cdots < y_1 < y$, then:

$$A(\alpha x, s, p) < A(\alpha y, t, p)\ .$$

<u>*Example 1*</u>. The discount function

$$A(c,t,p) = c \cdot e^{-\frac{k}{c}(t-p)}$$

verifies the magnitude effect. In effect, let us suppose that for all *x*, *y*, *s* and *t*, such that $x < y$ and $s < t$, $(x,s) \sim_p (y,t)$. Then

$$x \cdot e^{-\frac{k}{x}(s-p)} = y \cdot e^{-\frac{k}{y}(t-p)},$$

from where:

$$\frac{x}{y} = \frac{e^{-\frac{k}{y}(t-p)}}{e^{-\frac{k}{x}(s-p)}}\ .$$

This way,

$$A(\alpha x, s, p) = \alpha x \cdot e^{-\frac{k}{\alpha x}(s-p)} = \alpha x \left[e^{-\frac{k}{x}(s-p)} \right]^{\frac{1}{\alpha}} =$$

$$= \alpha x \left[\frac{y}{x} e^{-\frac{k}{y}(t-p)} \right]^{\frac{1}{\alpha}} = \alpha x \frac{y^{\frac{1}{\alpha}}}{x^{\frac{1}{\alpha}}} e^{-\frac{k}{y}(t-p)} =$$

$$= \alpha x^{1-\frac{1}{\alpha}} y^{\frac{1}{\alpha}} e^{-\frac{k}{y}(t-p)} < \alpha y^{1-\frac{1}{\alpha}} y^{\frac{1}{\alpha}} e^{-\frac{k}{y}(t-p)} = \alpha y \cdot e^{-\frac{k}{\alpha y}(t-p)}\ .$$

Let us see that, in this case, the necessary and sufficient condition shown in the previous theorem is verified. In effect,

$$\frac{\partial A(c,t,p)}{\partial c} = e^{-\frac{k}{c}(t-p)} \cdot \left[1 + \frac{k}{c}(t-p) \right]\ .$$

Then

$$\frac{\left.\frac{\partial A(c,s,p)}{\partial c}\right|_{c=x}}{\left.\frac{\partial A(c,t,p)}{\partial c}\right|_{c=y}} = \frac{e^{-\frac{k}{x}(s-p)}\left[1 + \frac{k}{x}(s-p)\right]}{e^{-\frac{k}{y}(t-p)}\left[1 + \frac{k}{y}(t-p)\right]} = \frac{y}{x} \frac{1 + \frac{k}{x}(s-p)}{1 + \frac{k}{y}(t-p)}\ .$$

On the other hand, as $\frac{e^{-\frac{k}{y}(t-p)}}{e^{-\frac{k}{y}(s-p)}} = \frac{x}{y} < 1$, then

$$\frac{k}{y}(t-p) > \frac{k}{x}(s-p),$$

from where:

$$\frac{1+\frac{k}{x}(s-p)}{1+\frac{k}{y}(t-p)} < 1.$$

Thus,

$$\frac{y}{x}\frac{1+\frac{k}{x}(s-p)}{1+\frac{k}{y}(t-p)} < \frac{y}{x}.$$

Certainly, if $\left.\frac{\partial A(c,s,p)}{\partial c}\right|_{c=x}$ is increasing along the indifference line, the necessary and sufficient condition of the previous theorem will be verified, although the aforementioned condition can be verified being the partial derivative decreasing.

In short, in theorems 4 to 6 we have stated some conditions for the discounting function including the magnitude effect which have been empirically found in several empirical studies revised at the beginning of this section.

The sign effect or gain-loss asymmetry

A different treatment can be observed in intertemporal choices of positive and negative results, that is to say, of gains and losses. The discount rates for losses are lower than the discount rates for gains; this has been called the sign effect or gain-loss asymmetry. So, we can observe, for example, a gain of 100 euros at the present moment equal to a gain of 200 euros in one year (discount rate of 100%), but also observe a loss of 100 euros at the present moment equal to a loss of 150 euros in a one year time (discount rate of 50%). Thaler (1981) shows in his empirical study that the discount rates applied to delays of 3 months, 1 year and 3 years in the payment of a fine were lower than the discount rates associated to comparable questions about monetary gains. Many subjects showed negative discounting, since they preferred an immediate to a delayed loss of the same magnitude.

Like the magnitude effect, the sign effect can be explained in terms of the value function for money. Prelec & Loewenstein (1991) proposed the *amplification* loss property that implies that, changing the sign of an amount from gains to losses, the weight of this amount increases; that is, the ratio of subjective values for losses is higher than the ratio of equivalent gains.

Prelec & Loewenstein (1991) formulate the sign effect as follows:

$$(x,s) \sim (y,t) \text{ implies } (-x,s) \succ (-y,t) \text{ for } y > x > 0 \text{ and } s < t .$$

The improving sequence effect

The improving sequence effect consists in the preference for sequences of results increasing over time, being demonstrated that preferences for sequences of results are, often, different from individual results choices. Thus, for individual results it is shown a positive time preference, whereas for sequences of results it is shown a negative time preference. This preference for improving sequences has been shown in several empirical studies (Loewenstein, 1987; Loewenstein & Sicherman, 1991; Loewenstein & Prelec, 1991; Chapman, 1996; Chapman, 2000) for monetary results as for non-monetary results (hedonic experiences and health consequences). However, Chapman (1996, 2000) introduces the influence of expectations about health and money evolution on their preferences for sequences of results in both fields. Thus, in the short term, decision makers prefer increasing sequences of both money and health because they expect to improve their position over time and, hence, they show a negative time preference. For example, experiments in Chapman (2000) showed that people preferred improving sequences of headache pain, where pain decreased over time. However, for long (lifetime) sequences, they continue preferring increasing sequences of money (negative time preference), but they prefer decreasing sequences of health (positive time preference), since most people expect to experience health that decreases as they age and not the contrary. This is exemplified in Chapman's experiments by the preference of respondents for declining sequences of athletic ability and declining sequences of improvement of facial wrinkles.

So, Chapman shows that preferences for sequences of outcomes depend on both the domain (health or money) and the length of the sequence. These quantitative differences in the discount of different categories of goods constitute what has been called *framing effect* (Lázaro *et al.*, 2000).

Focusing on short sequences, Loewenstein & Prelec (1991) showed that 80% of subjects preferred to have dinner at a fancy French restaurant in one month than in two months; that is, they preferred the best result sooner than later. However, the preferences changed when the French dinner was composed into a sequence with the Greek dinner: 57% preferred dinner at the Greek restaurant in one month and dinner at the French restaurant in two months, rather than the two dinners in inverse order. The subjects now did not prefer the more attractive result as soon as possible. But, they showed a slightly preference for a sequence with the best result delayed until the end of the sequence.

While in individual intertemporal choices it is shown a positive time preference (that is, a preference for the best result sooner rather than later), in the intertemporal choices between sequences it is, often, shown an apparently negative time preference, that is, a preference for increasing sequences (improving sequences). An explanation of the preference for improving sequences is that decision makers anticipate the adaptation to their current position in the sequence and, for loss aversion, they are adverse to decrease in their position (Chapman, 2001).

We could define the improving sequence effect as follows: For all *s* and *t*, with $s < t$, there is a c_0 big enough such that, for all $y > x > c_0$, it is verified that:

$$\{(x,s),(y,t)\} \succ_p \{(y,s),(x,t)\}$$

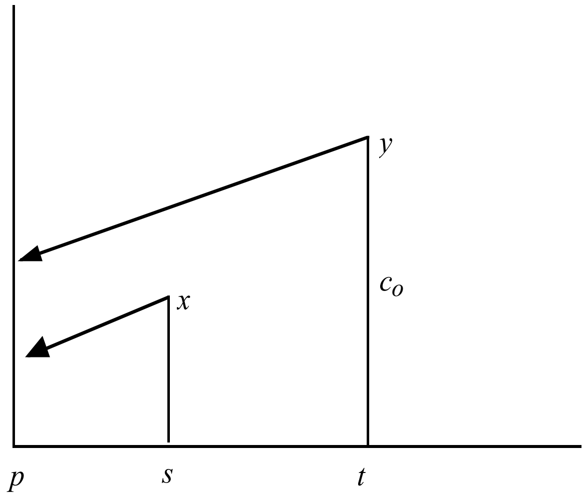

Figure 1. (*x*, *s*) and (*y*, *t*) projections onto *p*.

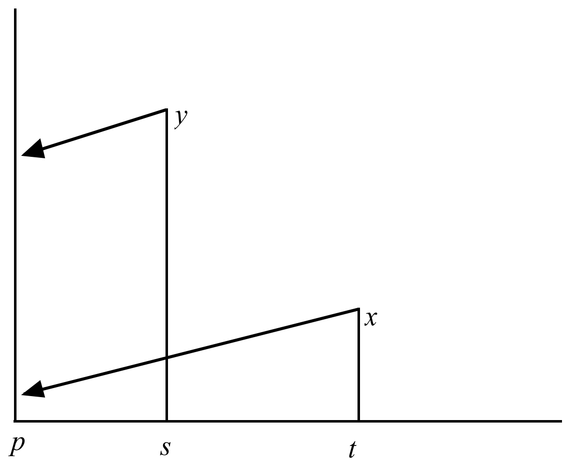

Figure 2. (*y*, *s*) and (*x*, *t*) projections onto *p*.

Let us consider, for example two amounts x and y, such that $x < y$ (x could represent the Greek dinner and y, the French one, in Loewenstein's example). If $A(t,p)$ is a discount function at p and $s < t$, then, being A strictly decreasing with respect to t, it must be verified that:

$$A(s,p) > A(t,p),$$

which implies:

$$A(s,p) - A(t,p) > 0.$$

As $x < y$, then it is verified that:

$$x\lfloor A(s,p) - A(t,p)\rfloor < y\lfloor A(s,p) - A(t,p)\rfloor,$$

$$xA(s,p) - xA(t,p) < yA(s,p) - yA(t,p),$$

from which, transposing terms:

$$xA(s,p) + yA(t,p) < yA(s,p) + xA(t,p),$$

that is to say,

$$\{(x,s),(y,t)\} \prec_p \{(y,s),(x,t)\}.$$

In other words, independently on the preference relation between the financial capitals (x,s) and (y,t) –even verifying that (x,s) is preferable at p to (y,t)– the previous presence of a higher amount capital, states the preference direction. Therefore, if the discount function used in the capital appraisal is homogeneous of the first degree with respect to the amounts, decreasing sequences will be always preferred to increasing ones, so it will be preferable the combination of two capitals with the higher amount capital available before in time. However, in real life these preferences change of direction. Thus, in Loewenstein & Sicherman's empirical research (1991), eighty subjects of those polled have to choose between increasing, decreasing or constant payment sequences (all options involved the same undiscounted total payoffs but differed in slope). In table 1, we can see these payment sequences and we can observe that the higher net present value corresponds to the decreasing sequence. But, inconsistently with the financial logic, the increasing sequence was the option chosen for most of the polled, concretely, the most chosen option was the payment sequence of work 5, that reflects an "intermediate" growth, if we consider the five increasing sequences.

All income sequences assign the same total amount, but with different installments for every option (jobs 1, 2, …, 7). The sequence with greater present value is the decreasing one, which will be a generalization (in this case to a sequence of six capitals) of what it has been explained about the preference of two capitals combination with the greater amount capital available before in time.

However, the inquiry results show a preference for increasing payment sequences (83% of respondents), contrary to what is predicted by the conventional theory of discount and showing the existence of improving sequence effect that implies a negative time preference in the case of positive outcome sequences.

Then, Loewenstein & Sicherman (1991) show that only a minority of inquired subjects showed some preferences consistent with the maximization of the present value. Most of them preferred an increasing sequence of payments that, as a whole, did not offer the highest present value, corresponding this maximum value to the decreasing sequence of payments. Even after the exposure to the arguments that justified the option for a decreasing sequence of payments, the majority of respondents continued to favor increasing payments. That is to say, the subjects showed a negative time preference (which implies a negative discount) that was justified for most of them by the pleasure of experiencing an increasing payment and consumption stream. There were other motivations like the compensation of decline in the standard of living due to the inflation, the pleasure of anticipating future consumption and the aversion to decreases in income or consumption and future spending needs that, in short, are all related with the former.

Table 1. Yearly income offered by different jobs (in thousands of dollars)

	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	PV*
Job 1	27	26.2	25.4	24.6	23.8	23	120.8
Job 2	25	25.0	25.0	25.0	25.0	25	119.8
Job 3	24	24.4	24.8	25.2	25.6	26	119.2
Job 4	23	23.8	24.6	25.4	26.2	27	118.7
Job 5	22	23.2	24.4	25.6	26.8	28	118.2
Job 6	21	22.6	24.2	25.8	27.4	29	117.6
Job 7	20	22.0	24.0	26.0	28.0	30	117.1

*Present value assuming an annual discount rate of 10%.

Source: Loewenstein and Sicherman (1991).

Following with our reasoning, the existence of the sequence effect, that is, the preference for increasing instead of decreasing sequences, leads us to search some necessary conditions for the existence of this type of preference. Really, in case of preference for increasing sequences, it should be verified that:

$$A(x,s,p) + A(y,t,p) > A(y,s,p) + A(x,t,p),$$

from which, transposing terms:

$$A(x,s,p) - A(x,t,p) > A(y,s,p) - A(y,t,p).$$

Dividing both members of the former inequality by (t – s) > 0:

$$\frac{A(x,s,p) - A(x,t,p)}{t-s} > \frac{A(y,s,p) - A(y,t,p)}{t-s}.$$

Taking now limits of the two former quotients, when (t – s) $\rightarrow$ 0, it would remain:

$$-\frac{\partial A(x,t,p)}{\partial t} > -\frac{\partial A(y,t,p)}{\partial t},$$

or, which is the same,

$$\delta(x,t,p) > \delta(y,t,p) \ ,$$

that is, the instantaneous discount rate used in the appraisal of discount functions is decreasing with respect to *c*, that will imply, among other things, that the discount function will not be homogeneous of degree one with respect to the amounts.

We can formulate the following theorem:

Theorem 7. A necessary and sufficient condition for the sequence effect is that the instantaneous discount rate is decreasing with respect to the amount.

Proof. Let x and y be two amounts such that $x < y$ and *s* and *t* two instants such that $s < t$. We want to show that

$$A(x,s,p) + A(y,t,p) > A(y,s,p) + A(x,t,p).$$

That is the same to show:

$$e^{-\int_p^s \delta(x,z,p)dz} + e^{-\int_p^t \delta(y,z,p)dz} > e^{-\int_p^s \delta(y,z,p)\,dz} + e^{-\int_p^t \delta(x,z,p)dz}$$

and equivalent to show that:

$$e^{-\int_p^t \delta(y,z,p)dz} - e^{-\int_p^s \delta(y,z,p)dz} > e^{-\int_p^t \delta(x,z,p)dz} - e^{-\int_p^s \delta(x,z,p)dz}$$

and this is verified since:

$$e^{-\int_p^s \delta(y,z,p)dz}\left[e^{-\int_s^t \delta(y,z,p)\,dz} - 1\right] > e^{-\int_p^s \delta(x,z,p)dz}\left[e^{-\int_s^t \delta(x,z,p)dz} - 1\right].$$

Let us see an example. Suppose that the instantaneous discount rate is inversely proportional to the square of the discounted amount:

$$\delta(c,z) = \frac{i}{c^2 - iz}, \ i > 0.$$

In this case, the corresponding discount function is:

$$A(c,z) = c\left(1 - \frac{iz}{c^2}\right).$$

In the Table 2 we can see the preference for increasing sequences, instead of decreasing ones, using this discount function for appraisal. We have taken $x = 20$, $y = 30$ and $i = 0.05$, with different *s* and $t = s + 2$ and $p = 0$.

Coming back to Loewenstein and Sicherman's example (1991), we can calculate the deviations d_t with respect to the total utility for each one of the options (works 1 to 7) and to include these deviations in the percentage of discount, so the new model will include the improving sequence effect. Thus, using a 10% discount rate, like Loewenstein & Sicherman did, the new discount rate will be $0.10 - \sum d_t$. If we calculate the net present value (NPV) for every option, but using this new discount rate, we can prove that the option with a higher NPV is work 7, since it shows the sequence with the biggest growth.

Table 2. Present value of two sequences (increasing and decreasing) at different time instants.

s	$A(x,s,p)+A(y,t,p)$	$A(y,s,p)+A(y,t,p)$
1	49.9925	49.9908
2	49.9883	49.9867
3	49.9842	49.9825
4	49.9800	49.9783
5	49.9758	49.9742
6	49.9717	49.9700
7	49.9675	49.9658
8	49.9633	49.9617
9	49.9592	49.9575
10	49.9550	49.9533
11	49.9508	49.9492
12	49.9467	49.9450
13	49.9425	49.9408
14	49.9383	49.9367
15	49.9342	49.9325
16	49.9300	49.9283
17	49.9258	49.9242
18	49.9217	49.9200
19	49.9175	49.9158
20	49.9133	49.9117

The spreading effect

The spreading effect is also referred to sequences of results and shows the subject's preferences for outcomes evenly spread over time. In their empirical research, Loewenstein & Prelec (1993) found that the 84% of subjects preferred to have dinner in French restaurant the second, instead of the first, of a three weeks period, when it was specified that they will eat at home the other weekends. However, when it was specified that they will have lobster dinner at a four-stars restaurant the third weekend, the 54% of subjects preferred to have dinner in French restaurant the first weekend, instead of the second. That is to say, the preference for having dinner in French restaurant before or after was conditioned by the event occurring the third weekend. Thus, it is shown that decision makers prefer to distribute the outcomes more evenly over time intervals. This pattern violates the additive separability that the discounted utility model implies.

In the same way we did for the improving sequence effect, we can include in Loewenstein & Sicherman's example (1991) the spreading effect. Using the same value for the discount rate, 10%, as the used one in the mentioned example, we will have only to add $\sum |d_t|$, as a component that measures the uniformity of the sequence, and now the discount rate will be $0.10 + \sum |d_t|$.

With this criterion, the more preferred work will be, now, the second one, since its sequence of payments is the most uniform.

The delay-speedup asymmetry

The delay-speedup asymmetry implies higher discount rates for decisions involving to delay rewards than for decisions involving to speedup them, which supposes an anomaly consisting in the asymmetric preference between speedup and delay consumption. Loewenstein (1988) has documented this effect, showing that the corresponding discount rates could be dramatically influenced by the change in the reward giving time being formulate as speedup or delay from some reference time moment. That way, in their experiment, the subjects who did not wait an immediate consumption, specifically a video camera record, will pay an average of 54 dollars to immediately receive it (instead of in one year), but those who thought that they would receive it immediately ask for an average of 126 dollars to delay for one year its receipt. Benzion, Rapaport & Yagil (1989) and Shelley (1993) obtained the same results as Loewenstein, either for losses as for gains (the polled asked for an amount bigger in the case of speeduping the payment than in the case of delaying it).

Conclusions

In this paper, it has been highlighted the existence of several anomalies in the performance of the axioms of traditional discounting models (DU and EU models), showed in empirical intertemporal choices made under uncertainty and the need to include their effects in the future discount models. More specifically, it has been observed inconsistent behavior with the predictions of traditional models in several empirical psychological researches in which people had to choose between alternatives with the delay or the amount modified in a significant quantity. Moreover, in these experiments, the results have been grouped in sequences and the decisions have been set out in terms of loss or gain or in terms of delay or speedup in the reception of a certain reward. This way, the different arisen anomalies have been labelled as delay effect, magnitude effect, improving sequence effect and spreading effect, sign effect and delay-speedup asymmetry. Every one of them has an effect in the choice of delayed rewards that has influence in discount rates and, consequently, in the respective discount models to be used. We have presented the empirical evidence of anomalies in intertemporal choice, describing them and stating some consequences of these anomalies in the discount functions, being this paper, then, a first attempt to design a discount model able to capture all the aforementioned anomalies and which could be modified according to the decision terms (decision about a sequence of outcomes, outcomes of the same or different sign, etc.). Further research is needed to offer a definitive discount model collecting all these effects that constitute anomalies for the traditional discount functions.

NOTES:

[1] Subadditive discounting means that the overalll discounting is higher when the interval is divided in installments and implies a smaller overall discount function for more subdivided intervals. A discount function will be subadditive if, for all *p, s, t,* with $p \leq s \leq t$, it is verified: $A(t,p) \geq A(t,s) \cdot A(s,p)$.

References

Ainslie, G. (1975). Specious reward: A behavioral theory of impulsiveness and impulse control. *Psychological Bulletin, LXXXII,* 463-509.

Ainslie, G. (1992). *Picoeconomics: The strategic interaction of successive motivational states within the person*. Cambridge: Cambridge University Press.

Azfar, O. (1999). Rationalizing hyperbolic discounting. *Journal of Economic Behavior* & *Organization, 38*, 245-252.

Angeletos, G., Laibson, D., Repetto, A., Tobacman, J. & Weinberg, S. (2001). The hyperbolic consumption model: Calibration, simulation, and empirical evaluation. *Journal of Economic Perspectives, 15,* 47-68.

Benzion, U., Rapaport, A. & Yagil, J. (1989). Discount rates inferred from decisions: An experimental study. *Management Science, 35,* 270-284.

Chapman, G. (1996). Expectations and preferences for sequences of health and money. *Organizational Behavior and Human Decision Processes, 67,* 59-75.

Chapman, G. (2000). Preferences for improving and declining sequences of health outcomes. *Journal of Behavioral Decision Making, 13,* 203-218.

Chapman, G. (2003). Time discounting of health outcomes. In G. Loewenstein, D. Read, & R. Baumeister (Eds.). *Time and decision: economic and psychological perspectives on intertemporal choice* (pp. 395-418). New York: Russell Sage Foundation.

Christensen-Szalanski (1984). Discount functions and the measurement of patients' values: Women's decisions during childbirth. *Medical Decision Making, 4,* 47-58.

Green, L., Fristoe, N. & Myerson, J. (1994). Temporal discounting and preference reversals in choice between delayed outcomes. *Psychonomic Bulletin and Review, 3*, 383-389.

Green, L., Fry, A. & Myerson, J. (1994). Discounting of delayed rewards: A life-span comparison. *Pychological Science, 5*, 33-36.

Green, L. & Myerson, J. (1996). Exponential versus hyperbolic discounting of delayed outcomes: risk and waiting time. *American Zoologist, 36*, 496-505.

Green, L., Myerson, J. & Ostazewski, (1999). Discounting of delayed rewards across the life span: Age differences in individual discounting functions. *Behavioural Processes, 46*, 89-96.

Harvey, C. (1986). Value functions for infinite-period planning. *Management Science, XXXII*, 1123-39.

Henderson, N. & Bateman, I. (1995). Empirical and public choice evidence for hyperbolic social discount rates and the implications for intergenerational discounting. *Environmental and Resource Economics, 5*, 413-423.

Herrnstein, R. (1981). Self-control as response strength. In C.M. Bradshaw, E. Szabadi & C.F. Lowe (Eds.), *Quantification of Steady State Operant Behavior* (pp. 3-20). Amsterdam: Elsevier/North-Holland Biomedical Press.

Holcomb, J. & Nelson, P. (1989). *An experimental investigation of individual time preference*. Unpublished Manuscript. University of Texas.

Kagel, J. Green, L. & Caraco, T. (1986). When foragers discount the future: constraint or adaptation? *Animal Behavior, 34*, 271-283.

Kirby, K. (1997). Bidding on the future: evidence against normative discounting of delayed rewards. *Journal of Experimental Psychology: General, 126*, 54-70.

Kirby, K. & Herrnstein, R. (1995). Preference reversals due to myopic discounting of delayed reward. *Psychological Science, 6*, 83-89.

Kirby, K. & Marakovic, N. (1995). Modeling myopic decisions: Evidence for hyperbolic delay-discounting within subjects and amounts. *Organizational Behavior and Human Decision Processes, 64*, 22-30.

Koopmans (1960). Stationary ordinal utility and impatience. *Econometrica, 28*, 287-309.

Laibson (1997). Golden eggs and hyperbolic discounting. *The Quarterly Journal of Economics, 112*, 443-477.

Laibson, D., Repetto, A. & Tobacman, J. (1998). Self-control and saving for retirement. *Brooking Papers on Economic Activity, 1*, 91-196.

Lázaro, A., Barberán, R. & Rubio, E. (2001). *El modelo de utilidad descontada y las preferencias sociales. Análisis de algunas formulaciones alternativas al descuento convencional*. III Encuentro de Economía Aplicada, Valencia.

Loewenstein, G. (1987). Anticipation and the valuation of delayed consumption. *Economic Journal, 97*, 666-684.

Loewenstein, G. (1988). Frames of mind in intertemporal choice. *Management Science, 34*, 200-214.

Loewenstein, G. & Prelec, D. (1991). Negative time preference. *American Economic Review, 81*, 347-352.

Loewenstein, G. & Prelec, D. (1992). Anomalies in intertemporal choice: Evidence and an interpretation. *The Quarterly Journal of Economics, CVII*, 575-597.

Loewenstein, G. & Prelec D. (1993). Preferences for sequences of outcomes. *Psychological Review, 100*, 91-108.

Loewenstein, G., Read, D. & Baumeister, R. (2003). Introduction. In G. Loewenstein, D. Read, & R.

Baumeister (Eds.), *Time and decision: economic and psychological perspectives on intertemporal choice* (pp. 1-12). New York: Russell Sage Foundation.

Loewenstein, G. & Sicherman, N. (1991). Do workers prefer increasing wage profiles? *Journal of Labor Economics, 9*, 67-84.

Loewenstein, G. & Thaler, R. (1989). Anomalies: intertemporal choice. *Journal of Economic Perspectives, 3*, 181-193.

Loewenstein, G. & Elster, J. (1992). *Choice over time*. New York: Russell Sage Foundation.

Mazur, J.E. (1987). An adjusting procedure for studying delayed reinforcement. In M.L. Commons, J.E. Mazur, J.A. Nevins & H. Rachlin (Eds.), *Quantitative analyses of behavior: Vol. 5. The effect of delay and of intervening events on reinforcement value* (pp. 55-73). Hillsdale, NJ: Erlbaum.

Myerson, J. & Green, L. (1995). Discounting of delayed rewards: Models of individual choice. *Journal of Experimental Analysis of Behavior, 64*, 263-276.

Myerson, J., Green, L. & Warusawitharana, M. (2001). Area under the curve as a measure of discounting. *Journal of Experimental Analysis of Behavior, 76*, 235-243.

Prelec, D. (1989). *Decreasing impatience: Definition and consequences*. Harvard Business School (working paper).

Prelec, D. & Loewenstein G. (1991). Decision making over time and under uncertainty: A common approach. *Management Science, 37*, 770-786.

Rachlin, H. (1989). *Judgement, decision and choice: A cognitive/behavioral synthesis*. New York: Freeman.

Rachlin, H., Raineri, A. & Cross, D. (1991). Subjective probability and delay. *Journal of the Experimental Analysis of Behavior, 64*, 263-273.

Read, D. & Loewenstein, G. (2000). Time and decision: introduction to the special issue. *Journal of Behavioral Decision Making, 13*, 141-144.

Richards, J., Mitchell. S., De Wit, H. & Seiden, L. (1997). Determination of discount functions in rats with an adjusting-amount procedure. *Journal of the Experimental Analysis of Behavior, 67*, 353-366.

Rodríguez, M. & Logue, A. (1988). Adjusting delay to reinforcement: Comparing choice in pigeons and human. *Journal of Experimental Psychology: Animal Behavior Processes, 14*, 105-117.

Samuelson, P. (1937). A note on measurement of utility. *Review of Economic Studies, 4*, 155-161.

Shelley, M. (1993). Outcome signs, questions frames and discount rates. *Management Science, 39*, 806-815.

Thaler, R. (1981). Some empirical evidence on dynamic inconsistency. *Economic Letters, 8*, 201-207.

Von Neumann, J. & Morgenstern, O. (1953). *Theory of games and economic behavior*. Princeton: Princeton University Press.

Received January 8, 2004
Final acceptance March 26, 2004